NOTE ON B. KLEIN'S "DIRECT USE OF EXTREMAL PRINCIPLES IN SOLVING CERTAIN PROBLEMS INVOLVING INEQUALITIES"

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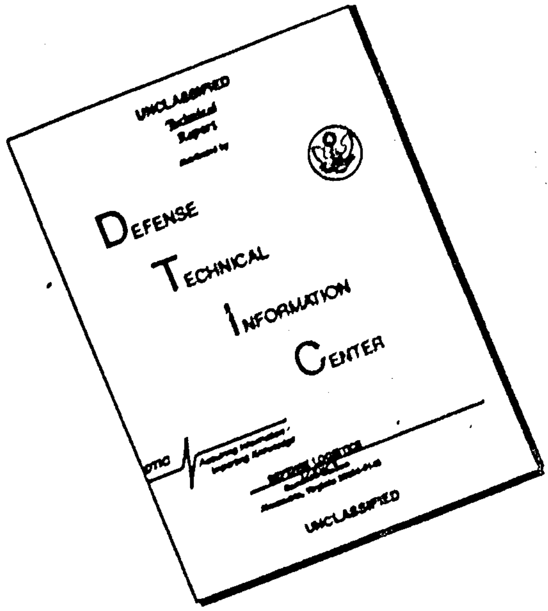

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SUMMARY

In the May, 1955 issue of the Journal of the Operations Research Society of America, B. Klein proposed that ordinary methods of the differential calculus be used to minimize a function z of n variables, x sub 1, x sub 2 . . . , x sub n [redacted] where the latter are subject to inequality constraints instead of the usual equality constraints. ~~Our purpose will be to~~ This report discusses whether this proposal can be used constructively to determine the optimum. (1)

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NOTE ON B. KLEIN'S "DIRECT USE OF EXTREMAL PRINCIPLES IN SOLVING CERTAIN PROBLEMS INVOLVING INEQUALITIES"

1. INTRODUCTION

In the May, 1955, issue of the Journal of the Operations Research Society of America, B. Klein [1] proposed that ordinary methods of the differential calculus be used to minimize a function z of n variables $x_1, x_2, \ldots, x_n$, where the latter are subject to inequality constraints instead of the usual equality constraints. Indeed, as a first step, by the introduction of new variables u_i he replaces all inequalities such as

$$(1) \quad f_i(x_1, x_2, \ldots, x_n) \leq 0$$

by equalities such as

$$(2) \quad f_i(x_1, x_2, \ldots, x_n) + u_i^2 = 0.$$

After this the problem can be solved (in theory) by the use of Lagrange multipliers. For the linear programming case the method yields the information that the solution is at a vertex where the sign of equality holds in some of the original constraints but perhaps not in others. For the nonlinear case it shows in general only that the solution is either inside or on the boundary. Our purpose will be to discuss whether this method of analysis can be used constructively to determine the optimum.

2. APPLICATION TO LINEAR PROGRAMMING PROBLEMS

For the standard linear programming problem

$$(3)\qquad \sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1,2,\ldots,m),$$

$$(4)\qquad x_j > 0 \qquad (j = 1,2,\ldots,n),$$

$$(5)\qquad \sum_{j=1}^{n} c_jx_j = z \qquad \text{(minimum)}$$

the inequality conditions (4) are replaced by

$$(4')\qquad -x_j + u_j^2 = 0 \qquad (j = 1,2,\ldots,n).$$

Let the Lagrange multipliers associated with the i-th equation of (3) and the j-th equation of (4') be π_i and δ_j, respectively. Then the main problem is reduced to that of finding the unconstrained minimum of the function

$$(6)\qquad z' = \sum_{j=1}^{n} c_jx_j - \sum_{i=1}^{m} \pi_i\left(\sum_{j=1}^{n} a_{ij}x_j - b_i\right) + \sum_{j=1}^{n} \delta_j(-x_j + \ldots$$

where the constants π_i and δ_j are selected so that x_j and u_j at the minimum satisfy (3) and (4'). Setting $\partial z'/\partial x_j = 0$ and $\partial z'/\partial u_j = 0$ yields <u>necessary</u> conditions for a minimum:

$$(7)\qquad c_j - \sum_{i=1}^{m} \pi_i a_{ij} - \delta_j = 0 \qquad (j = 1,2,\ldots,n),$$

$$(8)\qquad \delta_j u_j = 0,$$

or—solving for δ_j in (7) and substituting in (8)—

$$(9)\qquad \left(c_j - \sum_{i=1}^{m} \pi_i a_{ij}\right) u_j = 0 \qquad (j = 1,2,\ldots,n),$$

which are well-known relations that must exist between optimal solutions of the primal and dual linear programming problems.

The essence of Klein's proposal is that (9) implies for each j that either the first or the second factor must be zero. This gives rise to 2^n possible ways that (9) can be satisfied. Consider, for instance, the possibility that for $j_1, j_2, \ldots, j_r$ the first factor of the product vanishes and for all other j the second vanishes. The $n - r$ conditions $u_j = 0$ (i.e., $x_j = 0$) for $j \neq j_\nu$ when substituted into (3) give rise to m equations (10), below, in r variables x_j where $j = j_\nu$, and the r conditions $\delta_j = 0$ for $j = j_\nu$ give rise to r equations (11) in m variables π_i:

$$(10) \quad \sum_{\nu=1}^{r} a_{ij_\nu} x_{j_\nu} = b_i \qquad (i = 1,2,\ldots,m),$$

$$(11) \quad \sum_{i=1}^{m} \pi_i a_{ij_\nu} = c_j \qquad (\nu = 1,2,\ldots,r).$$

If $r < m$, this would result in the first system (10) having fewer equations than variables; if $r > m$, the second system (11), whose matrix is the transpose of the first, would have this property. Thus in practice it may be expected that each member of a large subset of the 2^n combinations will result in an inconsistent system. For example, if one could assume that the rank of any $m \times m$ submatrix of the coefficients of (1) is m, then all cases $r > m$ or $r < m$ could be dropped; thus instead of having to handle 2^n combinations, one could deal with the $\binom{n}{m}$ combinations where $r = m$. To avoid the practical difficulty

of testing the assumption on rank, one could introduce "artificial" variables as is usually done in linear programming and show that it is sufficient to consider cases $r = m$ with submatrices of rank m.

In the case of a linear programming problem for which values x_j exist satisfying (3) and (4) and for which there is a finite lower bound to the value of z, the procedure as outlined is completely valid. One simply selects, from among the combinations, where $n-m$ of the x_j are set equal to zero, solutions in which all x_j satisfy $x_j \geq 0$ and finds one that minimizes the value of z.

3. COMPUTATIONAL CONSIDERATIONS

It has been observed that many people who have considered problems involving inequalities have gone over the same ground as Klein, which we have just discussed. The practical difficulty arises only if the solution of $\binom{n}{m}$ systems, each $m \times m$, is too much work. Thus if $n = 7$, $m = 2$ as in the illustrative example presented by Klein, all combinations can be enumerated quickly. George Stigler [2], on the other hand, in trying to solve a nutrition problem involving 77 foods and 9 nutritional elements, made a search among the combinations of an ($n = 77$, $m = 9$) linear-programming nutrition problem. Actually, he was able to eliminate many foods and needed to consider an ($n = 23$, $m = 9$) case; since $\binom{23}{9}$ was also quite large, he had to resort to further devices to decrease the number of combinations to about 512. Even after this reduction, since this was done in the days of hand methods, he was forced to try a few of the remaining combinations and take for his "optimum" the best among them.

4. THE SIMPLEX METHOD

The simplex method [3] in fact begins where Klein leaves off. The first part of the procedure consists of a method of selecting, from among the $\binom{n}{m}$ combinations, a solution—called a basic feasible solution—in which all x_j satisfy $x_j \geq 0$ (the m variables $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$ are called basic variables).* With such a starting solution, the values of the π_i are determined next by (11) and of the δ_j by (7). It can be shown that all δ_j should satisfy $\delta_j \geq 0$ for the solution to be optimum. Thus, if there is a $\delta_s < 0$, one can allow the corresponding nonbasic variable $x_s = 0$ to increase while all other nonbasic variables remain at value zero. At a value $x_s = x_s^*$ sufficiently large one of the values of the basic variables becomes $x_{i_o} = x_{i_o}^* = 0$ while all other x_{j_ν} satisfy $x_{j_\nu} \geq 0$. This gives rise to a second combination in which x_s replaces x_{i_o} as basic variable. The new value of z can be shown to be less than that for the previous combination. The procedure is iterated until a solution satisfying (3) and (4) is obtained in which all δ_j satisfy $\delta_j \geq 0$. Approximately m combinations are usually examined in practice. The shift from one combination to the next can be arranged so that no more than mn new multiplications are needed. Thus each new combination does not involve solving from scratch a new m x m system of equations.

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*The values of the n-m nonbasic variables are zero.

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